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CHROMATIC POLYNOMIALS OF MIXED HYPERCYCLES

Julian A. Allagan

Department of Science & Mathematics
University of North Georgia
Watkinsville, Georgia

e-mail: julian.allagan@ung.edu

and

David Slutzky

Department of Science & Mathematics
University of North Georgia
Watkinsville, Georgia

e-mail: david.slutzky@ung.edu

Abstract

We color the vertices of each of the edges of a $\mathcal{C}-$hypergraph (or cohypergraph) in such a way that at least two vertices receive the same color and in every proper coloring of a $\mathcal{B}-$hypergraph (or bihypergraph), we forbid the cases when the vertices of any of its edges are colored with the same color (monochromatic) or when they are all colored with distinct colors (rainbow). In this paper, we determined explicit formulae for the chromatic polynomials of $\mathcal{C}-$hypercycles and $\mathcal{B}-$hypercycles.

Keywords: hypercycle, mixed hypergraph, chromatic polynomial.

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1. Introduction and Definitions

For basic definitions and terminology we refer the reader to [2, 6, 18]. A hypergraph $\mathcal{H}$ of order n is an ordered pair $\mathcal{H}=(X,\mathcal{E})$, where $|X| = n$ is a finite nonempty set of vertices and $\mathcal{E}$ is a collection of not necessarily distinct non empty subsets of X called (hyper)edges. $\mathcal{H}$ is said to be $k-$*uniform*, if the size of each of its edges is exactly k. A hypergraph is said to be *linear* if each pair of edges has at most one vertex in common. The *degree* of a vertex v is the number of edges containing v. A *hyperleaf* is a hyperedge which contains exactly one vertex of degree 2. In this paper all hypergraphs are assumed to be connected, linear and $k-$uniform unless stated otherwise. A linear *hypercycle* of length l is a hypergraph induced by a set of edges $\{e_1, \ldots, e_l\}$ $(l \geq 3)$ where

$$|e_i \cap e_j| = \begin{cases} 1 & if\ j = i+1\ or\ \{i,j\} \in \{1,l\} \\ 0 & otherwise. \end{cases}$$

We note that the term *elementary* hypercycle has also been used for *linear* hypercycle by Tomescu [15]. A (linear) hypercycle of length 2 induced by the set of edges $\{e_1, e_2\}$ can be defined where $|e_1 \cap e_2| = 2$. In the case where $l = 2$ and $k = 2$ we allow for a loop, but our results are concerned with $k > 2$ where the hypercycle of length 2 is a meaningful example. (See Example 1.1).

An *$l-$unicyclic hypergraph* $\mathcal{H}$=$(X, \mathcal{E})$ is a hypergraph in which there is exactly one set $\{e_1, \ldots, e_l\}$ which induces a hypercycle. A hypergraph which does not contain a hypercycle as a subhypergraph is called *acyclic*.

The concept of mixed-hypergraph coloring has been studied extensively by Voloshin et al. [9, 10, 18]. A *mixed hypergraph* $\mathcal{H}$ with vertex set X is a triple $(X, \mathcal{C}, \mathcal{D})$ such that $\mathcal{C}$ and $\mathcal{D}$ are subsets of X, called $\mathcal{C}-$(hyper)edges and $\mathcal{D}-$(hyper)edges, respectively. Elements of $\mathcal{C} \cap \mathcal{D}$ are called $\mathcal{B}-$(hyper)edges (or bi-edges). A proper coloring of $\mathcal{H}$ is a coloring of X such that each $\mathcal{C}-$edge has at least two vertices with a $\mathcal{C}$ommon color and each $\mathcal{D}-$edge has at least two vertices with $\mathcal{D}$istinct colors. Given the mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, when $\mathcal{C} = \emptyset$, we write $\mathcal{H} = (X, \mathcal{D})$ and call it a $\mathcal{D}-$hypergraph (or *hypergraph*). In the case when $\mathcal{D} = \emptyset$, we write $\mathcal{H} = (X, \mathcal{C})$ and call the mixed hypergraph a $\mathcal{C}-$hypergraph (or *cohypergraph*). In the case when $\mathcal{C} = \mathcal{D}$, we write $\mathcal{H} = (X, \mathcal{B})$ and call it a $\mathcal{B}-$hypergraph (or *bihypergraph*). Several important results and open problems about mixed hypergraphs and bihypergraphs can be found in [7, 8, 11, 12, 13, 14].

Example 1.1. A hypercycle of length 2.

Let $\mathcal{H}_2^3$=$(X, \mathcal{E})$ where $X = \{v_1, v_2, v_3, v_4\}$ and $\mathcal{E} = \{e_1, e_2\}$ with $e_1 = \{v_1, v_2, v_3\}$ and $e_2 = \{v_1, v_2, v_4\}$. Then Figure 1 is a representation of $\mathcal{H}_2^3$.

The *chromatic polynomial* $P(\mathcal{H}, \lambda)$ of a mixed hypergraph $\mathcal{H}$ is the function that counts the number of proper $\lambda-$colorings, which are mappings, $f : X \to \{1, 2, \ldots, \lambda\}$ with the condition that every $\mathcal{C}-$edge has at least two vertices with a $\mathcal{C}$ommon color and every $\mathcal{D}-$edge has at least two vertices with $\mathcal{D}$istinct colors. We encourage the reader to refer to [9, 10, 18] for detailed information about chromatic polynomials, research, and applications of mixed hypergraph colorings.

For simplicity, throughout this paper, we will denote by $\mathcal{H}_l^k = (X, \mathcal{E})$, a linear $k-$uniform hypergraph of length l, where $|\mathcal{E}|$=l. We also denote the falling

v_1

$P(\Pi^k_{l-1}, \lambda)$ proper colorings of Π^k_{l-1}, there exist $\lambda^{k-1} - (\lambda - 1)^{(k-1)}$ colorings of $X(e_1) \setminus v$ in which not all vertices have distinct colors to $f(v)$. This produces all $(\lambda^{k-1} - (\lambda - 1)^{(k-1)})P(\Pi^k_{l-1}, \lambda) = \lambda(\lambda^{k-1} - (\lambda - 1)^{(k-1)})^l = \lambda(\zeta(1))^l$ proper colorings of Π^k_l.

■

Remark 2. Most of the formulas in this paper are proven with an inductive argument similar to that of Theorem 1. We leave those inductive arguments to the reader and will indicate the place for the argument by ending subsequent proofs with, "the result follows by induction on l."

Theorem 3. *Let $\Pi^k_l = (X, \mathcal{B})$ be a $k-$uniform linear connected acyclic $\mathcal{B}-$hypergraph of length l. Then $P(\Pi^k_l, \lambda) = \lambda(\zeta(1) - 1)^l$.*

Proof. Consider $l = 1$. There are λ^k ways to color each of its vertices while exactly λ assign the same color to all vertices and $\lambda^{(k)}$ assign different colors to all k vertices of Π^k_l. Hence there are exactly $\lambda^k - \lambda^{(k)} - \lambda = \lambda(\lambda^{k-1} - (\lambda - 1)^{(k-1)} - 1)$ ways to color the edge so that not all of its vertices are either colored with the same or with different colors. The result follows by induction on l. ■

Theorem 4. *Let $\Pi^k_l = (X, \mathcal{D})$ be a $k-$uniform linear connected acyclic $\mathcal{D}-$hypergraph. Then $P(\Pi^k_l, \lambda) = \lambda(\zeta(1) + \gamma(1) - 1))^l$.*

Proof. Consider the case when $l = 1$ and name the edge e. There are λ^k ways to color each of its vertices while exactly λ assign the same color to all vertices, bringing the number of proper λ-colorings to $\lambda^k - \lambda = \lambda(\lambda^{k-1} - 1)^1$. The result follows by induction on l. ■

Corollary 1. Let $\Pi^k_l = (X, \mathcal{C}, \mathcal{D})$ be a $k-$uniform linear connected acyclic mixed hypergraph. Then $P(\Pi^k_l, \lambda) = \lambda\,(\gamma(1))^{p_1}\,(\zeta(1) - 1)^{p_2}\,(\zeta(1) + \gamma(1) - 1)^{p_3}$ where $|\mathcal{C} - \mathcal{D}| = p_1$, $|\mathcal{B}| = |\mathcal{C} \cap \mathcal{D}| = p_2$ and $|\mathcal{D} - \mathcal{C}| = p_3$.

Proof. The result follows from induction on $l = p_1 + p_2 + p_3$, by first considering the edges of $\mathcal{C} - \mathcal{D}$, then the edges of $\mathcal{B}$, and finally the edges of $\mathcal{D} - \mathcal{C}$. ■

3. THE CHROMATIC POLYNOMIALS OF SOME CYCLIC HYPERGRAPHS OF LENGTHS 2 AND 3

Theorem 5. *Let $\mathcal{H}^k_2 = (X, \mathcal{C})$ be a $k-$uniform $C-$hypercycle of length 2. Then*

$$P(\mathcal{H}^k_2, \lambda) = \lambda^{n-1} + \lambda^{(2)}(\zeta(2))^2.$$

Proof. Let $\mathcal{H}_2^k = (X, \mathcal{C})$ be a $k-$uniform linear hypercycle induced by the set of edges $\{c_1, c_2\}$. Consider their two vertices of degree 2, say, v_1 and v_2. In each proper coloring of $\mathcal{H}_2^k$, one of the following is true.

(i) $f(v_1) = f(v_2)$.

There are λ ways to color both vertices. Then the remaining $k-2$ vertices of each edge can be properly colored in λ^{k-2} ways. Hence the number of colorings is

$$\lambda(\lambda^{k-2})^2. \tag{1}$$

(ii) $f(v_1) \neq f(v_2)$.

There are $\lambda(\lambda-1)$ different ways to color both vertices. But there are $(\lambda^{k-2} - (\lambda-2)^{(k-2)})^2$ ways to color the remaining vertices of each edge, giving the number of colorings

$$\lambda(\lambda-1)(\lambda^{k-2} - (\lambda-2)^{(k-2)})^2. \tag{2}$$

By combining 1 and 2 we obtain

$$P(\mathcal{H}_2^k, \lambda) = \lambda^{2k-3} + \lambda(\lambda-1)(\lambda^{k-2} - (\lambda-2)^{(k-2)})^2 \tag{3}$$

as desired.

■

Theorem 6. *Let $\mathcal{H}_2^k = (X, \mathcal{B})$ be a $k-$uniform $B-$hypercycle of length 2. Then*

$$P(\mathcal{H}_2^k, \lambda) = \lambda\left(\zeta(2) + \gamma(2) - 1\right)^2 + \lambda^{(2)}(\zeta(2))^2.$$

Proof. This proof is very similar to the one in Theorem 5. Let $\mathcal{H}_2^k = (X, \mathcal{B})$ be a $k-$uniform linear hypergraph induced by the set of edges $\{b_1, b_2\}$. Consider their two vertices of degree 2, say, v_1 and v_2. In each proper $\lambda-$coloring of $\mathcal{H}_2^k$, one of the following is true.

(i) $f(v_1) = f(v_2)$.

There are λ ways to color both vertices. Then the remaining $k-2$ vertices of each edge can be properly colored in $\lambda^{k-2} - 1$ ways. Hence the number of colorings is

$$\lambda(\lambda^{k-2} - 1)^2. \tag{4}$$

(ii) $f(v_1) \neq f(v_2)$.

There are $\lambda(\lambda-1)$ different ways to color both vertices. But there are $(\lambda^{k-2} - (\lambda-2)^{(k-2)})^2$ ways to color the remaining vertices of each edge, giving the number of colorings

$$\lambda(\lambda-1)(\lambda^{k-2} - (\lambda-2)^{(k-2)})^2. \tag{5}$$

By combining 4 and 5 we obtain that

$$P(\mathcal{H}_2^k, \lambda) = \lambda(\lambda^{k-2} - 1)^2 + \lambda(\lambda-1)(\lambda^{k-2} - (\lambda-2)^{(k-2)})^2. \tag{6}$$

■

Example 2.1. A proper 3-coloring of a linear 3-uniform $\mathcal{C}-$hypercycle of length 3.

Let $\mathcal{H}_3^3$=$(X,\mathcal{C})$ where $X = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $\mathcal{C} = \{c_1, c_2, c_3\}$ with $c_1 = \{v_1, v_3, v_4\}$, $c_2 = \{v_1, v_2, v_5\}$ and $c_3 = \{v_2, v_3, v_6\}$. Figure 2 is a representation of $\mathcal{H}_3^3$, a linear 3-uniform hypercycle of length 3. Letting for instance $f(v_1) = f(v_4) = 1$, $f(v_2) = f(v_5) = 2$, and $f(v_3) = f(v_5) = 3$, we have a proper 3-coloring of $\mathcal{H}_3^3$.

(ii) Two colors are used to color these three vertices.

Suppose $f(v_1) \neq f(v_2) = f(v_3)$. Then there are $\lambda(\lambda-1)$ ways to color the three vertices. Now there are λ^{k-2} ways to color the remaining vertices of c_3 (which does not contain v_1) while there are $(\lambda^{k-2}-(\lambda-2)^{(k-2)})^2$ ways to color the remaining vertices of c_1 and c_2 (which contain v_1). Since there are three different ways of choosing the one vertex of different color, the number of colorings is

$$3\lambda(\lambda-1)(\lambda^{k-2})(\lambda^{k-2}-(\lambda-2)^{(k-2)})^2. \tag{8}$$

(iii) All three vertices v_1, v_2, v_3, have different colors.

There are $\lambda(\lambda-1)(\lambda-2)$ different ways to color the three vertices. There are $(\lambda^{k-2}-(\lambda-2)^{(k-2)})^3$ ways to color the remaining vertices of each edge. The total number of colorings in this case is

$$\lambda(\lambda-1)(\lambda-2)(\lambda^{k-2}-(\lambda-2)^{(k-2)})^3. \tag{9}$$

Now we combine (7), (8), and (9) to obtain the desired result. ■

Theorem 8. *Let $\mathcal{H}_3^k = (X, \mathcal{B})$ be a $k-$uniform $B-$hypercycle of length 3. Then* $P(\mathcal{H}_3^k, \lambda) = \lambda\left(\zeta(2)+\gamma(2)-1\right)^3 + 3\lambda^{(2)}\left(\zeta(2)+\gamma(2)-1\right)\left(\zeta(2)\right)^2 + \lambda^{(3)}(\zeta(2))^3$.

Proof. Using similar steps as in the proof of Theorem 7, we obtain that $\lambda(\lambda^{k-2}-1)^3+3\lambda(\lambda-1)(\lambda^{k-2}-1)(\lambda^{k-2}-(\lambda-2)^{(k-2)})^2+\lambda(\lambda-1)(\lambda-2)(\lambda^{k-2}-(\lambda-2)^{(k-2)})^3$, giving the desired result.

■

The chromatic polynomials of mixed hypergraphs are often computed using a recursive algorithm, commonly known as splitting-contraction [18]. To derive an explicit form for such formulas using the splitting-contraction algorithm is at least ♯P-hard. However, using some combinatorial and recursive arguments, we obtained some (albeit not so simple) forms of these polynomials. These generalized formulas are presented in the next section and are built on the chromatic polynomials of the cyclic mixed hypergraphs already discussed in this section.

4. Chromatic polynomials of cyclic hypergraphs of arbitrary length

Theorem 9. *Let $\mathcal{H}_l^k = (X, \mathcal{D})$ be a $k-$uniform $\mathcal{D}-$hypercycle. Then* $P(\mathcal{H}_l^k, \lambda) = (\lambda-1)^l \left(\sum_{i=0}^{k-2} \lambda^i\right)^l + (-1)^l(\lambda-1)$ *for all* $l \geq 2$.

One of the authors proved this theorem in [1] and it has been established independently by Borowiecki and łazuka, as Walter pointed out in [19], simply because $(\lambda-1)^l(\sum_{i=0}^{k-2}\lambda^i)^l+(-1)^l(\lambda-1) = \left((\lambda-1)\sum_{i=0}^{k-2}\lambda^i\right)^l+(-1)^l(\lambda-1) = (\lambda^{k-1}-1)^l+(-1)^l(\lambda-1)$.

A considerable amount of literature has been written concerning the chromatic polynomials of certain families of $\mathcal{D}-$hypergraphs by Borowiecki et al. and Tomescu et al., just to name a few researchers [3, 4, 5, 15]. However, very little is known about these formulas as they relate to mixed hypergraphs in general, particularly, the $\mathcal{C}-$hypergraphs and $\mathcal{B}-$hypergraphs. We present here some new results about these particular members of mixed hypergraphs.

Theorem 10. *Let $\mathcal{H}_l^k = (X, \mathcal{C})$ be a $k-$uniform $C-$hypercycle of length $l \geq 3$. Then*

$$P(\mathcal{H}_l^k, \lambda) = \zeta(2)P(\Pi_{l-1}^k, \lambda) + \gamma(2)P(\mathcal{H}_{l-1}^k, \lambda) \tag{10}$$

where Π_l^k is a $k-$uniform linear connected acyclic $\mathcal{C}-$hypergraph of length $l \geq 3$.

Proof. Let $\mathcal{H}_l^k = (X, \mathcal{E})$ be any $k-$uniform $C-$hypercycle of length $l \geq 3$ induced by the set of edges $\{c_1, \ldots, c_l\}$. Let u and v be the two vertices of degree 2 in c_l. In any proper coloring of the edge c_l using at most λ colors, either (i) u and v have the same color, or (ii) u and v have different colors. We therefore count the number of such colorings for each case in turn.

Case (i) There are λ^{k-2} ways to color the remaining $k-2$ vertices in $c_l\backslash\{u, v\}$ so that at least two vertices receive the same color, and there are $P(\mathcal{H}_{l-1}^k, \lambda)$ ways to color the remaining vertices so that $f(u) = f(v)$. Hence, there are $\lambda^{k-2}P(\mathcal{H}_{l-1}^k, \lambda)$ colorings.

Case (ii) Let Π_{l-1}^k be the hyperpath of length $l-1$ induced by $\{c_1, \ldots, c_{l-1}\}$. There are $\lambda^{k-2} - (\lambda-2)^{(k-2)}$ colorings of the vertices in $c_l\backslash\{u, v\}$. For each such coloring, the number of colorings of the remaining vertices is

$$P(\Pi_{l-1}^k, \lambda) - P(\mathcal{H}_{l-1}^k, \lambda),$$

since the first term counts the number of colorings where u and v may have the same or different colors, and the second term counts the number of colors where u and v have the same color. So there are

$$\left(\lambda^{k-2} - (\lambda-2)^{(k-2)}\right) P(\Pi_{l-1}^k, \lambda) + (\lambda-2)^{(k-2)}P(\mathcal{H}_{l-1}^k, \lambda)$$

colorings altogether.

■

Corollary 2. Let $\mathcal{H}_l^k = (X, \mathcal{C})$ be a $k-$uniform $C-$hypercycle of length $l \geq 3$. Then

$$P(\mathcal{H}_l^k, \lambda) = (\gamma(2))^{l-2}\lambda^{2k-3} + \lambda\zeta(2)\sum_{j=1}^{l-2}(\gamma(2))^{j-1}(\zeta(1))^{l-j} + \lambda^{(2)}(\zeta(2))^2(\gamma(2))^{l-2}.$$

Proof. When $l = 2$, the middle term is set to zero to yield $P(H_2^k, \lambda) = \lambda^{2k-3} + \lambda^{(2)}(\zeta(2))^2$, which becomes the basis of the recursive argument for the proof. When $l = 3$, the formula in Theorem 7 can be expanded (although messy) to support this result. Now, for $l \geq 3$, we obtain from (10) that

$$P(\mathcal{H}_l^k, \lambda) = (\gamma(2))^{l-2}P(\mathcal{H}_2^k, \lambda) + \zeta(2)\sum_{j=1}^{l-2}(\gamma(2))^{j-1}P(\Pi_{l-j}^k, \lambda).$$

Using Theorems 1 and 5, we obtain the result after substitution.

■

Theorem 11. *Let $\mathcal{H}_l^k = (X, \mathcal{B})$ be a $k-$uniform $B-$hypercycle of length $l \geq 3$. Then*

$$P(\mathcal{H}_l^k, \lambda) = \zeta(2)P(\Pi_{l-1}^k, \lambda) + (\gamma(2) - 1)\, P(\mathcal{H}_{l-1}^k, \lambda) \tag{11}$$

where Π_{l-1}^k is a $k-$uniform linear connected acyclic $\mathcal{B}-$hypergraph.

Proof. Let $\mathcal{H}_l^k = (X, \mathcal{B})$ be any $k-$uniform $B-$hypercycle of length l induced by the set of edges $\{b_1, \ldots, b_l\}$ $(l \geq 3)$. Let u and v be the 2 vertices of degree 2 in b_l. In any proper coloring of the edge b_l using $\lambda-$colors, either (i) u and v have the same color, or (ii) u and v have different colors. We therefore count the number of such colorings for each case in turn.

Case (i) There are $\lambda^{k-2} - 1$ ways to color the remaining $k - 2$ vertices in $b_l\backslash\{u, v\}$ so that at least two vertices (of the remaining $k - 2$ vertices) receive different colors, and there are $P(\mathcal{H}_{l-1}^k, \lambda)$ ways to color the remaining vertices so that $f(u) = f(v)$. Hence, there are $(\lambda^{k-2} - 1)P(\mathcal{H}_{l-1}^k, \lambda)$ colorings.

Case (ii) Let Π_{l-1}^k be the hyperpath of length $l - 1$ induced by $\{b_1, \ldots, b_{l-1}\}$. There are $\lambda^{k-2} - (\lambda - 2)^{(k-2)}$ colorings of the vertices in $b_l\backslash\{u, v\}$. For each such coloring, the number of colorings of the remaining vertices is

$$P(\Pi_{l-1}^k, \lambda) - P(\mathcal{H}_{l-1}^k, \lambda),$$

since the first term counts the number of colorings where u and v may have the same or different colors, and the second term counts the number of colors where u and v have the same color. So there are

$$\left(\lambda^{k-2} - (\lambda - 2)^{(k-2)}\right) P(\Pi_{l-1}^k, \lambda) + \left((\lambda - 2)^{(k-2)} - 1\right) P(\mathcal{H}_{l-1}^k, \lambda)$$

colorings altogether. ■

Corollary 3. Let $\mathcal{H}_l^k = (X, \mathcal{B})$ be a $k-$uniform $B-$hypercycle of length $l \geq 3$. Then $P(\mathcal{H}_l^k, \lambda) = \lambda(\zeta(2) + \gamma(2) - 1)^2(\gamma(2) - 1)^{l-2} + \lambda\zeta(2)\sum_{j=1}^{l-2}(\gamma(2) - 1)^{j-1}(\zeta(1) - 1)^{l-j} + \lambda^{(2)}(\zeta(2))^2(\gamma(2) - 1)^{l-2}$.

Proof. When $l = 2$, the middle term is set to zero to yield $P(\mathcal{H}_2^k, \lambda) = \lambda\left(\zeta(2) + \gamma(2) - 1\right)^2 + \lambda^{(2)}(\zeta(2))^2$, which becomes the basis of the recursive argument for the proof just as in the previous corollary. For $l \geq 3$, we obtain from (11) that $P(\mathcal{H}_l^k, \lambda) = (\gamma(2) - 1)^{l-2}P(\mathcal{H}_2^k, \lambda) + \zeta(2)\sum_{j=1}^{l-2}(\gamma(2) - 1)^{j-1}P(\Pi_{l-j}^k, \lambda)$. Using Theorems 3 and 6, we obtain the desired formula. ■

These results obtained in this section can easily be rewritten to obtain the chromatic polynomials of several other families of linear connected uniform hypergraphs. In particular the chromatic polynomials of unicyclic mixed hypergraphs and mixed hypercacti [9] can be written and are left as exercises for the reader. As it is, rewriting these formulas in terms of the standard basis is doable but messy. Further work could look for simpler forms for these expressions or address the remaining open problems of interpreting the coefficients of these polynomials and finding their roots.

Furthermore, by using γ and ζ as functions of $|e|$ (i.e., of any value other than just k), it is reasonable to extend the formulas discussed in this paper to non-uniform mixed hypergraphs (see Corollary 4). Recently, Walter [19] has found the formulas for some non-uniform $\mathcal{D}-$hypergraphs. As a step in this direction, we close this paper with a more general result concerning non-uniform acyclic mixed hypergraphs.

It is easy to verify that the chromatic polynomials of an isolated hyperedge, cohyperedge and bihypereredge are as follows.

Proposition 1. Let e be an isolated hyperedge. Then the chromatic polynomials of e when viewed as a $\mathcal{D}-$hyperedge, $\mathcal{C}-$hyperedge, or $\mathcal{B}-$hyperedge are

$$\begin{aligned} P_{\mathcal{D}}(e) &= \lambda(\lambda^{|e|-1} - 1) \\ P_{\mathcal{C}}(e) &= \lambda(\lambda^{|e|-1} - (\lambda - 1)^{\underline{|e|-1}}) = \lambda\zeta_{|e|}(1) \\ P_{\mathcal{B}}(e) &= \lambda(\lambda^{|e|-1} - (\lambda - 1)^{\underline{|e|-1}} - 1) = \lambda(\zeta_{|e|}(1) - 1) \end{aligned} \tag{12}$$

respectively.

For instance, the case when $e \in \mathcal{D}$, there are $\lambda^{|e|} - \lambda = \lambda(\lambda^{|e|-1} - 1)$ ways to properly color each hyperedge.

From (12), we can extend Corollary 1 (following the argument used in Theorem 1) to obtain the following.

Corollary 4. Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ be an acyclic mixed hypergraph. Then the chromatic polynomial of any (non-uniform) acyclic mixed hypergraph is given by

$$P(\mathcal{H}) = \lambda \prod_{\substack{e_1 \in \mathcal{D}, e_2 \in \mathcal{C} \\ e_3 \in \mathcal{B}}} (\lambda^{|e_1|-1} - 1)\zeta_{|e_2|}(1)(\zeta_{|e_3|}(1) - 1).$$

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